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ON THE STRUCTURE OF THE PHASE SPACE AND BIFURCATIONS IN THE DISCRETE LORENZ MODEL OF CONVECTIVE TURBULENCE*

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Results of qualitative and numerical investigations of the Lorenz system of differential equations are presented. Analysis of that system is reduced to the consideration of point mapping of a straight line into a straight line. This provides a visual and comparatively simple geometric representation of the system motion, thus simplifying the analysis and providing more comprehensive information.

Study of the Lorenz system is based on the investigation of isoclinal structures and of the part played by these in the formulation of stochastic motions /1—6/. The use of numerical method yields a series of one-dimensional point mappings that geometrically interpret the structure of solutions of the Lorenz system and its variation as parameter r is increased. Existence of isoclinal structures which appear when integral manifolds of saddle-type periodic motions intersect equilibrium states is established. Changes of that structure produced by variation of parameter r is considered. It appears that generation of stochastic motions is associated with transition through a reasonalby smooth isoclinal structure. As r is increased, degradation of stochasticity takes place at which the phase structure rapidly changes with variation of r, and complex stable periodic motions with fine regions of attraction develop.

The Lorenz three-dimensional autonomous system of differential equations was investigated in a number of publications (e.g., /1,7—14/). The interest in it is due to that the Lorenz system is a Galerkin approximation of equations of two-dimensional convection and, also, to the unusual behavior of its solutions, which may be considered as some model of turbulence generation /1,8/. Other phenomena, for example, the process of oscillation generation in certain types of lasers /1/, are also defined by Lorenz equations. The Lorenz equations are interesting in themselves, as the subject of the qualitative theory of differential equations and of the theory of oscillations.

1. General information on the Lorenz system and its reduction to the study of point transformation. Consider the Lorenz system of differential equations

$$x^{\cdot} = -\sigma x + \sigma y, \quad y^{\cdot} = rx - y - xz, \quad z^{\cdot} = -bz + xy \tag{1.1}$$

where σ, b, r are positive parameters. Let us point out some singularities in the behavior of phase trajectories of system (1.1).

1°. It admits the substitution of variables $-x, -y, -z$ for x, y, z without affecting its form. Its phase trajectories are, consequently, symmetric about the z axis.

2°. It admits the system of spheres

$$V(x, y, z) = x^2 + y^2 + (z - \sigma - r)^2 = \text{const}$$

for which by virtue of (1.1)

$$V^{\cdot} = -2\sigma x^2 - 2y^2 - 2b\,[z - {}^1/_2\,(\sigma + r)]^2 + {}^1/_2 b\,(\sigma + r)^2$$

Then $V^{\cdot} < 0$ when $V > b(\sigma + r)^2$ $(b > 2,\ \sigma > 1)$ and, consequently, phase trajectories from the whole space x, y, z converge inside the sphere

$$x^2 + y^2 + (z - \sigma - r)^2 = (\sigma + r)^2 b \tag{1.2}$$

of radius $(\sigma + r)\sqrt{b}$ with center at point $(0, 0, \sigma + r)$.

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3°. By virtue of system (1.1) we have

$$\operatorname{div}(x^{\cdot}, y^{\cdot}, z^{\cdot}) = -1 - \sigma - b < 0 \tag{1.3}$$

Hence, as the time t increases, the phase volume in any place of the phase space undergoes an $\exp(1 + \sigma + b)t$ -fold contraction. The space volume decrease is not due to multidirectional compression, on the contrary, compression in some directions is accompanied by stretching in others.

Consider the equilibrium state of system (1.1). Equating to zero its right-hand sides, we find that when $r < 1$ there is the unique equilibrium state $O\ (x = y = z = 0)$, and when $r > 1$ there are three equilibrium states $O\ (x = y = z = 0)$, $O_{1,2}\ (x = y = \pm\sqrt{b\,(r-1)},\ z = r - 1)$. The characteristic equations of equilibrium states O and $O_{1,2}$ are, respectively, of the form

$$(\lambda + b)[\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r)] = 0 \tag{1.4}$$

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + \sigma(\sigma + r)\lambda + 2b\sigma(r - 1) = 0 \tag{1.5}$$

As in earlier investigations (e.g., /1/), we set $\sigma = 10$, $b = {}^8/_3$. It follows from (1.4) and (1.5) that as r increases from zero, the equilibrium states undergo the following transformations: the nodal equilibrium state O which is stable when $r < 1$, becomes bifurcated when $r = 1$, after which it becomes saddle unsteady; simultaneously two stable nodal equilibrium states O_1 and O_2 are generated from O, which with increasing r become successively stable nodes and foci, and when $r = 24.74$ are transformed into unstable saddle-foci. Note the singularity of behavior of phase trajectories: with increasing time, their rapid approach to some surfaces takes place. This behavior of trajectories, directly observable in numerical analysis, is in accord with the indicated above over-all intensive contraction of the phase volume, and with the considerable differences between the real parts of roots $\lambda_{1,2}$ and λ_3 of the characteristic equation (1.5), which are defined by the approximate formulas

$$\lambda_{1,2} = 0.037r - 0.915 \pm i\,(0.18r + 3.7)$$
$$\lambda_3 = -0.06r - 11.5\ (10 < r < 40)$$

The change of stability of saddle-foci O_1 and O_2 is accompanied by the merging with them of unstable saddle periodic motions Γ_1 and Γ_2 generated from loops of separatrices S_1^- of saddle O when $r = 13.92$.

When $r < 24.06$ the over-all behavior of phase trajectories is simple: with $0 < r < 1$ all of them move to the stable equilibrium state O, and for $1 < r < 24.06$ all phase trajectories, except the separating ones, move to one of the stable equilibrium states O_1 or O_2. The separating phase trajectories constitute a two-dimensional invariant surface S_2^+ of the saddle equilibrium states O.

When $r > 24.06$, the over-all behavior of phase trajectories is much more complicated; it was investigated in /1,8–14/. To study the behavior of phase trajectories in this case we use the method of point mapping T generated by trajectories of system (1.1) of the plane $z = r - 1$ into itself. We denote that intersecting plane by Σ. Mapping T is discontinuous along lines K_1 and K_2 of contact of phase trajectories with the Σ plane, whose equation is of the form $xy = b\,(r - 1)$. It is also discontinuous on line R of intersection of the Σ plane with the separating surface S_2^+ of the saddle equilibrium state O. In some neighborhood of that line mapping T is smooth and can be continuously extended on R. We denote by T_1 and T_2 of narrowing of mapping T on different sides of line R, which are continuously prolonged onto the latter (Fig.1, a–b), where mapping T_1 is determined to the right of line R and T_2 to the left of it. These mappings transform line R into intersection points M_1 and M_2 of the Σ plane with separatrices S_1^- of the equilibrium state O. Application of mappings T_1 and T_2 transforms points M_1 and M_2 into points M_1' and M_2', respectively. Numerical determination of coordinates of points M_1' and M_2' shows that, as parameter r is increased they approach line R and when passing through $r = 13.92$ simultaneously intersect R, moving from one side of it to the other. In the Σ plane fixed points denoted by the same symbols correspond to the equilibrium states O_1 and O_2.

As previously stated, all phase trajectories reach the sphere (1.2), with increasing time t. This enables us to confine the investigation of the point mapping T to the circle

$$x^2 + y^2 < b\,(\sigma + r)^2 - (\sigma + 1)^2 \tag{1.6}$$

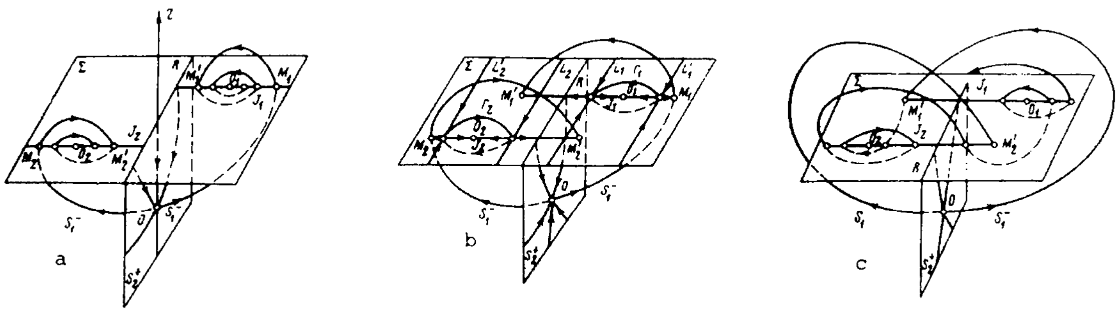

Fig.1

When $1 < r < 13.92$ this circle, after one mapping T, is transformed into some bounded bands of width not exceeding 0.05 and, after two such mappings into bands not wider than 10^{-3} (in Fig.1,a–c they are represented by lines J_1 and J_2). With further application of T mappings the width of these bands continues to decrease, the bands simultaneously shrink, and undergo the inward transformation into the stable fixed points O_1 and O_2. The latter conforms to the previously noted behavior of phase trajectories and roots of the characteristic equation of the equilibrium states O_1 and O_2.

When $r > 13.92$, the behavior of successive transformations of region (1.6) is more complex, since then points of regions inside circle (1.6) on different sides of line R can convert into each other. However even now successive transformations approach as rapidly some curves J_1 and J_2. The numerically calculated curves J_1 and J_2 with $r = 20, 24.4, 50, 80$ are shown respectively, in Fig.2,a–d.

Thus the T mapping is strongly compressive in directions toward curves J_1 and J_2. Hence the investigation of that mapping reduces to the mapping of curves J_1 and J_2 into themselves and into each other. To represent point mapping of curves J_1 and J_2 we select along them coordinate u as the distance from the discontinuity line R, and take it as positive in one direction and negative in the other. The numerically calculated successor functions $V = f(u)$ for the T^2 mapping of curves J_1 and J_2 are shown in Fig.3.

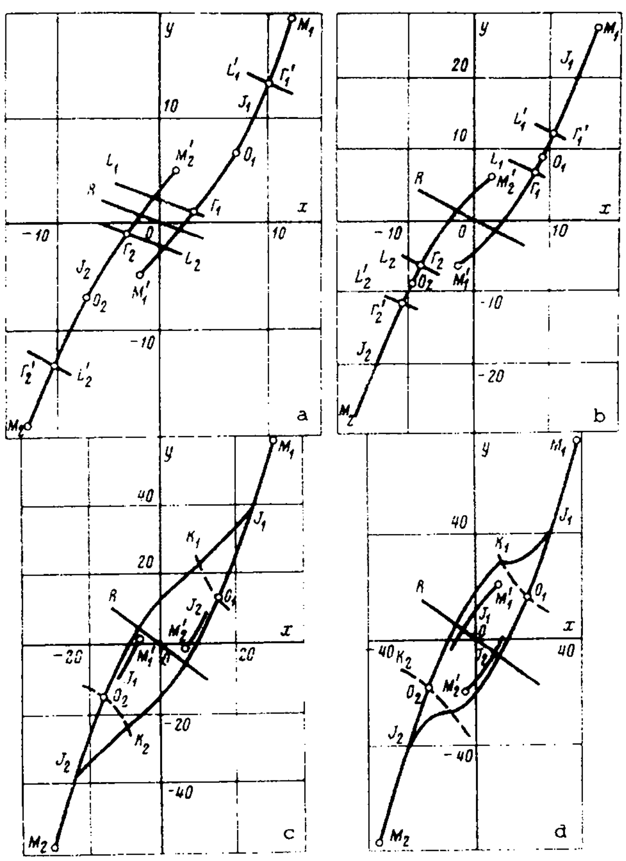

Fig.2

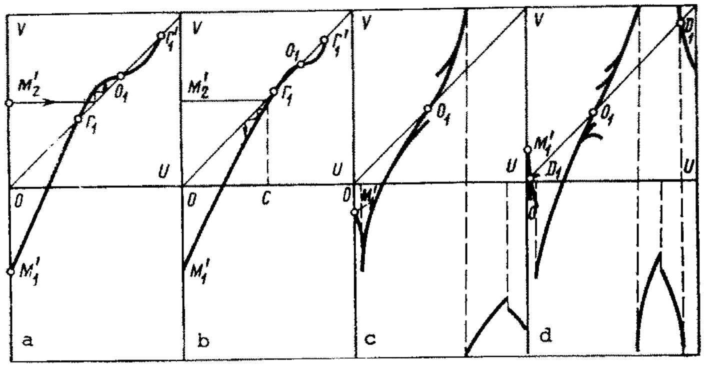

Fig.3

2. Point mapping and the phase space structure. Let us investigate the structure of the phase space x, y, z subdivision in system (1.1) by considering the dependence of the corresponding to it point mapping T on parameter r. As indicated in Sect.1, the equilibrium state O is over-all stable when $r < 1$ and, when $1 < r < 13.92$, all phase trajectories approach the equilibrium states O_1 and O_2 whose attraction regions are divided by the separating surface S_2^+ of the saddle equilibrium state O (see Fig.1,a). Below we consider the case of $r \geqslant 13.92$. When $r = 13.92$, points M_1 and M_2 of the second intersection of separatrices S_1^- with the Σ plane reach curve R, which corresponds to the formation of loops of separatrices S_1^-. When $r > 13.92$, the saddle periodic motions Γ_1 and Γ_2 are generated from these loops (Fig.1,b). In the separating plane Σ shown in Fig.2,a (for $r = 20$) the stationary double saddle points Γ_1, Γ_1' and Γ_2, Γ_2', for which lines J_1 and J_2, and L_1, L_1' and L_2, L_2' are, respectively, the outgoing and incoming invariant curves, correspond to motions Γ_1 and Γ_2. It will be seen that the invariant curves J_1, J_2 and L_1, L_2 intersect each other forming an isoclinal structure whose appearance in system (1.1) produces an infinite set of various saddle periodic and other motions in the neighborhood of the isoclinal structure /16/. However the behavior of successive iterations of the point mapping T upto $r = 24.06$ remains on the whole, simple: all points, except the set Ω of zero measure points, approach one of the stable fixed points O_1 or O_2. The set Ω plays the part of the boundary separating the attraction regions of these points.

The above is reflected in the behavior of the successor function $V = f(u)$ for curve J_1, shown in Fig.3,a, for the T^2 mapping of curves J_1 and J_2 (by virtue of the property 1° of the successor function for curve J_2 is symmetric to that for J_1 about the coordinate origin). The points of intersection with the bisectrix correspond to fixed points of the T mapping. Points M_1' and M_2' of the second intersection of separatrices S_1^- with the Σ plane lie on the V axis symmetrically about the coordinate origin, with point M_2' above point Γ_1 and point M_1' below point Γ_2. Hence the successive transformations of points M_1' and M_2' reach the regions of attraction of the stable fixed points O_1 and O_2, respectively.

The described structure of the T^2 mapping remains unchanged when $13.92 < r < 24.06$. When $r = 24.06$, points M_1' and M_2' reach the invariant curves L_1 and L_2, which corresponds to the appearance of a fairly smooth isoclinal structure. The coordinates of points M_2' and Γ_1 (M_1' and Γ_2) have different values on the curve of function $V = f(u)$. When $r > 24.06$, point M_2' drops below point Γ_1, and the intersection point of curves J_1 and L_1 vanish. There are three alternatives for the phase trajectories, viz., approach one of the equilibrium points O_1 or O_2, or continuously intersect the separating plane Σ. The relative position of the T mapping curves is shown in Fig.2,b for $r = 24.4$, and in Fig.3,b is shown the T^2 mapping curve of J_1. It will be seen that point M_2' lies below point Γ_1, hence successive iterations of point M_2' are on the segment $[-c, c]$ (c is the ordinate of point M_2'). Since on segment $-c, c]$ $f'(u) > 1$, the T^2 mapping is a stretching one. The iterated sequencies of points that begin on segment $[-c, c]$ correspond to stochastic motions of system (1.1).

When $r = 24.74$, the fixed double saddle points Γ_1, Γ_1' and Γ_2, Γ_2' merge with the stable fixed points O_1 and O_2, which corresponds to the merging of saddle periodic motions Γ_1 and

Γ_2 with the stable equilibrium states O_1 and O_2. As shown in /16/, in the case of such bifurcations the Liapunov quantity g_0 must be positive, which can be verified directly.

When $r > 24.74$, all points of the Σ plane in the T mapping reach the small neighborhoods of curves J_1 and J_2 (Fig.1,c). The attraction region of stochastic motions coincides with the entire space of system (1.1). As parameter r increases, the form of curves J_1 and J_2 and of their transformation into themselves and into each other undergoes a number of changes. As shown in Fig.2c,d, $r = 50$ and 80, respectively) each of curves J_1 and J_2 consists of two parts, which is due to the discontinuity of the T mapping of the dividing Σ into itself along lines K_1 and K_2 of contact of phase trajectories of system (1.1) with the Σ plane. Points M_1' and M_2' for $r = 31.05$. This is accompanied by the emergence of new points of the successor function discontinuity and complication of its curves (shown in Fig.3,c.d for $r = 50$ and 80, respectively). As r increases, points M_1' and M_2' again approach line R, and at $r = 54.65$ intersect it. This corresponds to the formation of new loops by separatrices S_1^- of the equilibrium state O. As in the case of $r = 13.92$, saddle periodic motions arise from these loops. The double fixed points D_1 and D_2' correspond to them in Fig.3,d.

Beginning with some $r > 65$ the point mapping curves have points at which the tangent is horizontal. This means that there can be no stochasticity in any arbitrarily small interval of values of r, in the sense that there exists everywhere a dense set of values of r that correspond to the existence of stable fixed multiple points. These stable points can also appear at lower values of r, when the stretching properties of mapping of curves J_1 and J_2 are lost, however, in the presence of points with horizontal tangents it is automatically so. Thus, when $r = 100$, stable periodic motions were disclosed; they correspond to the following six-term cycle of T mapping fixed points:

$$A_1(-17.12; -30.3), A_2(-11.62; -1.56), \quad A_3(-22.27; -37.28)$$
$$A_4(-9.32; 7.63), \quad A_5(27.7; 56.93), \quad A_6(2.62; -28.26)$$

This stable cycle exists in the interval $[99,98; 100,05]$.

The particular character of behavior of solutions of the Lorenz system for these values of parameter r may be defined as follows. Assume that some random perturbations with finite correlation time are added to the right-hand sides of equations. Then for each value of parameter r there exists some threshold of the perturbations magnitude, after which a random wandering begins in the neighborhood of the two-dimensional surface that corresponds to curves J_1 and J_2 on the dividing /surface/ Σ. For some r these threshold values are zero, for others they are small. Thus for the periodic motion disclosed when $r = 100$ the random wanderings appear in the form of random jerks not exceeding 0.05 of the x coordinate, occurring over time intervals $\Delta t = 4 \cdot 10^{-2}$.

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